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A Lebesgue–Lusin property for linear operators of first and second order

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We prove that for a homogeneous linear partial differential operator $\mathcal{A}$ of order $k \leqslant 2$ and an integrable map f taking values in the essential range of that operator, there exists a function u of special bounded variation satisfying

$$\mathcal{A}u(x) = f(x) \qquad \text{almost everywhere.}$$

This extends a result of G. Alberti for gradients on $\mathbf{R}^N$. In particular, for $0 \leqslant m < N$, it is shown that every integrable m-vector field is the absolutely continuous part of the boundary of a normal $(m+1)$-current.

Keywords: bounded variation; Lusin property; first-order; second-order; differential operator

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1. Introduction

Let Ω be an open subset of $\mathbf{R}^N$. We consider a general constant-coefficient system of linear partial differential equations acting on functions $u : \Omega \to E$, that is, a partial differential operator of the form

$$\mathcal{A}u = \sum_{|\alpha|=k} A_\alpha \partial^\alpha u, \qquad A_\alpha \in \mathrm{Hom}(E, F), \tag{1.1}$$

where E, F are finite-dimensional $\mathbf{R}$-spaces. Here $\alpha = (\alpha_1, \ldots, \alpha_N) \in (\mathbb{N}_0)^N$ is a multi-index with modulus $|\alpha| = \alpha_1 + \cdots + \alpha_N$ and $\partial^\alpha = \partial_1^{\alpha_1} \circ \cdots \circ \partial_N^{\alpha_N}$ is the composition of distributional directional partial derivatives. The Fourier transform establishes a one-to-one correspondence between homogeneous operators and their associated principal symbol $\mathbb{A} : \mathbf{R}^N \to \mathrm{Hom}(E, F)$, which in this context is given by the k-homogeneous tensor-valued polynomial

$$\mathbb{A}^k(\xi) = \sum_{|\alpha|=k} A_\alpha \xi^\alpha, \qquad \xi^\alpha = \xi_1^{\alpha_1} \cdot \cdots \cdot \xi_N^{\alpha_N}, \quad \xi \in \mathbf{R}^N.$$

Suppose now that $u \in \mathscr{D}'(\Omega; E)$ is a distribution such that $\mathcal{A}u$ is a zero-order distribution, i.e. represented by an F-valued Radon measure. Let $g : \Omega \to F$ be the polar vector of $\mathcal{A}u$. If, additionally, $\mathcal{A}$ is a constant-rank operator (see e.g. [8, 18, 19]), the

De Philippis–Rindler theorem [11, Theorem 1.1] establishes that g is constrained to take values on the *image cone* at *singular points*. More precisely, if $U \subset \Omega$ is a $\mathscr{L}_N$-negligible Borel set, where $\mathscr{L}_N$ stands for the N-dimensional Lebesgue measure, then

$$g(x) \in \mathcal{I}_{\mathcal{A}} := \bigcup_{|\zeta|=1} \operatorname{Im} \mathbb{A}^k(\zeta) \qquad |\mathcal{A}u| \text{ almost everywhere on } U.$$

This property extends a classical result of Alberti [3], which says that if the distributional gradient Du of a map $u : \mathbf{R}^N \to \mathbf{R}^M$ is represented by an $\mathbf{R}^{M\times N}$-valued Radon measure, then its polar vector must take values in the cone of rank-one matrices on $\mathscr{L}_N$-negligible sets: if $\mathscr{L}_N(U) = 0$, then

$$\operatorname{rank} \frac{Du}{|Du|}(x) = 1 \quad |Du| \text{ almost everywhere in } U.$$

In contrast with this restriction over the polar of Du at singular points, Alberti showed [1, Theorem 3] that the absolutely continuous part of a gradient measure is fully unconstrained: if $f \in L^1(\mathbf{R}^N; \mathbf{R}^{M\times N})$, then there exists $u \in BV(\mathbf{R}^N; \mathbf{R}^M)$ such that[1]

$$Du = f\,\mathscr{L}_N + [u] \otimes \nu_u\,\mathscr{H}_{N-1} \llcorner J_u, \qquad [u] := u^+ - u^-, \tag{1.2}$$

and

$$\|u\|_{BV} \leqslant C \int_\Omega |f|\,.$$

Alberti (see [1, 2]) also established *other* Lusin-type properties for gradients of arbitrary order. He showed that if $f : \Omega \to \mathbf{R}^{N^k}_{\mathrm{sym}}$ is continuous, then for any positive measure λ on Ω and any smallness constant $\epsilon > 0$, one may find a compact set $K \subset \Omega$ and a function $u \in C^k(\Omega)$ such that

$$f(x) = D^k u(x) \quad \forall x \in K\,, \qquad \lambda(\Omega \setminus K) \leqslant \epsilon\lambda(\Omega)\,,$$

and satisfying L^∞-estimates for $D^k u$ in terms of f. Building on these ideas, Francos [16] (see also [12]) showed that for any given Borel $f : \Omega \to \mathbf{R}^{N^k}_{\mathrm{sym}}$ and $\sigma > 0$, there exists a function $g \in C^{k-1}(\Omega)$, k-times differentiable almost everywhere, satisfying

$$f(x) = D^k g(x) \quad \text{a.e. on } \Omega \quad \text{and} \quad \|D^k g\|_{L^\infty(\Omega)} \leqslant \sigma.$$

Driven by applications to higher-order variational problems where derivatives give rise to surface energies, Fonseca et al. [15, Theorem 1.4] established an analogue of (1.2) for the Hessian operator. More precisely, they showed that if $f : \Omega \to \mathbf{R}^{N\times N}_{\mathrm{sym}}$, then there exists $u \in W^{1,1}(\Omega)$, with $\nabla u \in BV(\Omega; \mathbf{R}^N)$, satisfying

$$D^2 u = f\,\mathscr{L}_N + [\nabla u] \otimes \nu_u\,\mathscr{H}_{N-1} \llcorner J_{\nabla u} \quad \text{and} \quad \|u\|_{BH(\Omega)} \leqslant C\|f\|_{L^1}\,.$$

Unlike the Lusin-type properties for higher-order gradients by Francos, this generalization to second-order derivatives is not followed by a straightforward iteration

[1] Alberti's result is written originally for functions $f : \Omega \to \mathbf{R}$. Applying it on each component yields the general case for $\mathbf{R}^M$-valued vector fields.

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of Alberti's property (1.2). The reason for this drawback is the presence of the symmetry constraints of higher-order curl-free fields.

In light of this digression, the recent developments in the study of fine properties in $BV^{\mathcal{A}}$ spaces (see e.g. [**4**, **6**, **9**, **10**, **13**]), and the De Philippis–Rindler theorem, we are led to ask the following natural question: *is the absolutely continuous part of an $\mathcal{A}$-gradient measure fully unconstrained?* Our main results (Theorems 2.1 and 2.2) establish that, *at least* when $\mathcal{A}$ is an operator of order $k \leqslant 2$, this is indeed the case; regardless of $\mathcal{A}$ satisfying the constant-rank condition. Our proof hinges on Proposition 2.4, where we show the previous question is equivalent to the validity of (1.2) for Hessians of arbitrary order. This is the main reason why our result is restricted to the case $k = 1, 2$.

2. Notation and results

In all that follows, Ω will denote an open subset of $\mathbf{R}^N$. We will write $\mathscr{L}_N$ to denote the N dimensional Lebesgue measure and $\mathscr{H}_{N-1}$ to denote the $(N-1)$-dimensional Hausdorff outer measure on $\mathbf{R}^N$. The space $BV(\Omega; E)$, of E-valued vector fields with bounded variation on Ω, consists of all integrable maps $u : \Omega \to E$ whose distributional gradient can be represented by a finite Radon measure taking values on $E \otimes \mathbf{R}^N$. For such functions, we shall write $\nabla u \in L^1(\Omega; E \otimes \mathbf{R}^N)$ to denote the absolutely continuous part of Du with respect to $\mathscr{L}_N$. It is well known (see e.g. [**5**]) that such maps are Lebesgue continuous outside of a countably $\mathscr{H}_{N-1}$ rectifiable set $J_u \subset \Omega$, with orientation normal ν_u, called the jump set of u. Moreover, the map u has Lebesgue one-sided limits $u^+(x)$, $u^-(x)$ for $\mathscr{H}_{N-1}$ almost every $x \in J_u$ with respect to the normal direction $\nu_u(x)$, and their difference

$$[u](x) \coloneqq u^+(x) - u^-(x), \qquad x \in J_u,$$

defines a $\mathscr{H}_{N-1}$-integrable map on J_u. By the Radon–Nikodym theorem, the gradient of a map $u \in BV(\Omega; E)$ can be decomposed as $Du = \nabla u \, \mathscr{L}_N + D^s u$ where $D^s u \perp \mathscr{L}_N$. The singular part $D^s u$ can be further decomposed into mutually singular measures as

$$D^s u = D^c u + D^j u,$$

where (the *Cantor part*) $D^c u$ is a measure vanishing on countable unions of $\mathscr{H}_{N-1}$-finite Borel sets, and the so-called *jump part*

$$D^j u \coloneqq [u] \otimes \nu_u \, \mathscr{H}_{N-1} \llcorner J_u$$

is the restriction of Du to the set of jump discontinuities of u. The subspace $SBV(\Omega; E)$ of E-valued functions of special bounded variation consists of all functions $u \in BV(\Omega; E)$ with rectifiable singular gradient, i.e.

$$Du = \nabla u \, \mathscr{L}_N + [u] \otimes \nu_u \, \mathscr{H}_{N-1} \llcorner J_u \, .$$

We also define

$$\mathbf{F}_{\mathcal{A}} \coloneqq \overline{\{ \mathcal{A}u(x) : x \in \Omega, \, u \in \mathscr{D}(\mathbf{R}^N; E) \}}^F, \tag{2.1}$$

the *essential range* of the operator $\mathcal{A}$. It is easy to see that every distribution $\mathcal{A}u$ takes values in this space, which is the smallest space with this property.

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With these considerations in mind, we can state our main results. We begin by stating the result for first-order operators:

THEOREM 2.1. *Let $\mathcal{A}$ be a first-order operator as in (1.1) and let $f:\Omega\to\mathbf{F}_\mathcal{A}$ be integrable. Then there exists a map $u\in SBV(\Omega;E)$ satisfying*

$$\mathcal{A}u \;=\; f\,\mathscr{L}_N \;+\; \mathbb{A}^1(\nu_u)[u]\,\mathscr{H}_{N-1}\llcorner J_u \tag{2.2}$$

and

$$\int_\Omega(|u|+|\nabla u|)\,\mathrm{d}x+\int_{J_u}|[u]|\,\mathrm{d}\mathscr{H}_{N-1}\leqslant C\int_\Omega|f|\,\mathrm{d}x$$

for some constant C that only depends on $\mathcal{A}$.

A similar statement holds for second-order operators:

THEOREM 2.2. *Let $\mathcal{A}$ be a second-order operator as in (1.1) and let $f:\Omega\to\mathbf{F}_\mathcal{A}$ be integrable. Then there exists a map $u\in W^{1,1}(\Omega;E)$, with $\nabla u\in SBV(\Omega;E\otimes\mathbf{R}^N)$, satisfying*

$$\mathcal{A}u \;=\; f\,\mathscr{L}_N \;+\; \mathbb{A}^2(\nu_{\nabla u})(\,[\nabla u]\nu_{\nabla u}\,)\,\mathscr{H}_{N-1}\llcorner J_{\nabla u} \tag{2.3}$$

and

$$\int_\Omega(|u|+|\nabla u|+|\nabla^2u|)\,\mathrm{d}x+\int_{J_{\nabla u}}|[\nabla u]|\,\mathscr{H}_{N-1}\leqslant C\int_\Omega|f|\,\mathrm{d}x$$

where C is a constant that only depends on $\mathcal{A}$.

REMARK 2.3. The essential range $\mathbf{F}_\mathcal{A}$ defined in (2.1) coincides with the space $\operatorname{span}\{\,\mathbb{A}^k(\xi)[e]:\xi\in\mathbf{R}^N,\,e\in E\,\}=\operatorname{span}\{\mathcal{I}_\mathcal{A}\}$ of all $\mathcal{A}$-gradient amplitudes in Fourier space (for a proof see [7, Section 2.5]). For example, if $\mathcal{A}=D^2$ is the Hessian, then $\mathbf{F}_{D^2}=\operatorname{span}\{\,\xi\otimes\xi:\xi\in\mathbf{R}^N\,\}=:\mathbf{R}^{N\times N}_{\mathrm{sym}}$.

Theorems 2.1 and 2.2 will follow directly from the equivalence result below (Proposition 2.4), Alberti's original result for gradients (in the first-order case) and its analogue for Hessians by Fonseca, Leoni and Paroni (in the second-order case). Before stating Proposition 2.4, it will be convenient to introduce some basic notation for symmetric tensors and gradients of higher order.

Symmetric tensors

Let $r\geqslant 0$ be an integer. For a vector $v\in\mathbf{R}^N$, we write $v^{\otimes^r}$ to denote the tensor that results by taking the tensorial product of v with itself r-times (with the convention $v^{\otimes^0}=1$). We consider the subspace

$$\mathbf{R}^{N^r}_{\mathrm{sym}}=\operatorname{span}\left\{\,v^{\otimes^r}:v\in\mathbf{R}^N\,\right\}$$

consisting of all symmetric rth order tensors on $\mathbf{R}^N$. In the following, we shall consider a contraction $\langle\bullet,\,\bullet\rangle_r:(E\otimes\mathbf{R}^{N^r}_{\mathrm{sym}}\times\mathbf{R}^N)\to E$ associated with the canonical

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inner product $(\bullet, \bullet)$ on $\otimes^r \mathbf{R}^N$ as

$$\langle e \otimes V, v\rangle_r = e(V, v^{\otimes^r})\text{for all } e \in E, V \in \mathbf{R}^{N^r}_{\mathrm{sym}} \text{ and } v \in \mathbf{R}^N.$$

Notice that under our convention for $r = 0$, we have $\langle e, \bullet\rangle_0 = e$.

Jump densities for higher-order BV^k-spaces

Let $k \geqslant 2$ be an integer. We define the space $BV^k(\Omega)$ as the space of integrable functions $u : \Omega \to \mathbf{R}$ whose distributional k^{th} order gradient $D^k u$ can be represented by a Radon measure taking values on $\mathbf{R}^{N^k}_{\mathrm{sym}}$; the spaces $BV^k_{\mathrm{loc}}(\Omega)$, $BV^k(\Omega; E)$ are defined accordingly with this definition in the obvious manner. By classical elliptic regularity theory, there is a natural continuous embedding $BV^k_{\mathrm{loc}}(\Omega; E) \hookrightarrow W^{k-1,1}_{\mathrm{loc}}(\Omega; E)$. Therefore, the distributional gradient $D^{k-1}u$ of a map u in $BV^k_{\mathrm{loc}}(\Omega; E)$ can be represented by an integrable map $\nabla^{k-1}u : \Omega \to E \otimes \mathbf{R}^{N^{k-1}}_{\mathrm{sym}}$, that is,

$$D^{k-1}u = \nabla^{k-1}u\, \mathscr{L}_N.$$

Moreover, in this case, the tensor field $w \coloneqq \nabla^{k-1}u$ has bounded variation, the jump set J_w of w is a countably $\mathscr{H}_{N-1}$ rectifiable set where w has approximate one-sided limits $[w] \in E \otimes \mathbf{R}^{N^{k-1}}_{\mathrm{sym}}$, with respect to a fixed orientation ν_w of J_w, and the difference map $[w]$ is integrable on J_w. Since $D^k u$ is a symmetric-valued tensor measure, it follows that the $\mathscr{H}_{N-1}$ density of the jump part of $Dw = D^k u$ is of the form

$$[w] \otimes \nu_w = \lambda_w \otimes (\nu_w)^{\otimes^k}$$

for some $\mathscr{H}_{N-1}$-integrable map $\lambda_w : J_w \to E$. Using that $|\nu_w|^2 = 1$, we can characterize the E-coordinate coefficient λ_w directly in terms of $[w]$ and the contraction pairing defined above. Indeed, $\langle [w], \nu_w\rangle_{k-1} \cdot |\nu_w|^2 = \langle [w] \otimes \nu_w, \nu_w\rangle_k = \lambda_w |\nu_\omega|^{2k} = \lambda_w$. In particular, it follows that

$$[w] = \lambda_w \otimes (\nu_w)^{\otimes^{k-1}} \qquad \mathscr{H}_{N-1} \text{ almost everywhere on } J_w. \tag{2.4}$$

For consistency, we set $\nabla^0 u \coloneqq u$ so that, when $k = 1$, we simply get

$$[w] = \lambda_w = [u]\mathscr{H}_{N-1} \text{ almost everywhere on } J_u.$$

With these considerations in mind, we are now in the position to state the following equivalence between Lebesgue–Lusin properties for kth order gradients and arbitrary homogeneous operators of order k:

PROPOSITION 2.4. *The following statements are equivalent:*

(a) *Let $\mathcal{A}$ be a k^{th} order linear operator as in (1.1) and let $f : \Omega \to \mathbf{F}_{\mathcal{A}}$ be an integrable map. Then there exists $u \in BV^k(\Omega; E)$, satisfying*

$$\mathcal{A}u \;=\; f\,\mathscr{L}_N \;+\; \mathbb{A}^k(\nu_w)\langle [w], \nu_w\rangle_{k-1}\, \mathscr{H}_{N-1} \llcorner J_w\,,$$

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where $w := \nabla^{k-1}u$. *Moreover,* $w \in SBV(\Omega; E \otimes \mathbf{R}^{k-1}_{\mathrm{sym}})$ *and*

$$\|u\|_{W^{k-1,1}(\Omega)} + \int_\Omega |\nabla w| \,\mathrm{d}x + \int_{J_w} |[w]| \,\mathrm{d}\mathscr{H}_{N-1} \leqslant C \int_\Omega |f| \,\mathrm{d}x$$

for some constant C *depending on* $\mathcal{A}$.

(b) *Let* $f : \Omega \to \mathbf{R}^{N^k}_{\mathrm{sym}}$ *be an integrable map. Then there exists* $u \in BV^k(\Omega)$ *satisfying*

$$D^k u \;=\; f\,\mathscr{L}_N \;+\; [w] \otimes \nu_w\, \mathscr{H}_{N-1} \llcorner J_w,$$

where $w := \nabla^{k-1}u$. *Moreover,* $w \in SBV(\Omega; \mathbf{R}^{N^{k-1}}_{\mathrm{sym}})$ *and*

$$\|u\|_{W^{k-1,1}(\Omega)} + \int_\Omega |\nabla w| \,\mathrm{d}x + \int_{J_w} |[w]| \,\mathrm{d}\mathscr{H}_{N-1} \leqslant C \int_\Omega |f| \,\mathrm{d}x$$

for some constant C *depending on* N *and* k.

Proof of Proposition 2.4. The implication (a) $\Longrightarrow$ (b) is straightforward from the following observations. Firstly, the principal symbol of the kth order Hessian is given by $D^k(\xi) = \xi^{\otimes^k}$. Therefore, D^k has the form (1.1) with $E = \mathbf{R}$ and $F = \mathbf{R}^{N^k}_{\mathrm{sym}}$, by setting $A_\alpha = M_\alpha$, where the family $\{\, M_\alpha : |\alpha| = k \,\}$ is the canonical orthonormal basis of $\mathbf{R}^{N^k}_{\mathrm{sym}}$. Hence, from (2.4) and (a) we conclude that $u \in BV^k(\Omega)$ satisfies

$$D^k u = f\,\mathscr{L}_N + [w] \otimes \nu_w\, \mathscr{H}_{N-1} \llcorner J_w,$$

with $w = \nabla^{k-1}u \in SBV(\Omega; \mathbf{R}^{N^{k-1}}_{\mathrm{sym}})$, satisfying

$$\|u\|_{W^{k-1,1}(\Omega)} + \int_\Omega |\nabla w| \,\mathrm{d}x + \int_{J_w} |[w]| \leqslant C(D^k) \int_\Omega |f| \,\mathrm{d}x.$$

This proves the first implication since D^k implicitly fixes the spatial dimension N.

We now show (b) $\Longrightarrow$ (a). The first step is to use the alternative jet expression $A[D^k u] = \mathcal{A}u$, where $A : E \otimes \mathbf{R}^{N^k}_{\mathrm{sym}} \to \mathbf{F}_{\mathcal{A}}$ is the (unique) linear map satisfying

$$A[e \otimes \xi^{\otimes^k}] = \mathbb{A}^k(\xi)[e] \qquad \text{for all } e \in E\,. \tag{2.5}$$

The existence of A is a direct consequence of the universal property of the tensor product and the k-linearity of the principal symbol on the frequency variable ξ. For the sake of simplicity, let us write $\mathbf{E}_k = E \otimes \mathbf{R}^{N^k}_{\mathrm{sym}}$. By construction, we have

$$\operatorname{Im} A = \operatorname{span}\left\{\, \mathbb{A}^k(\xi)[e] : \xi \in \mathbf{R}^N, e \in E \,\right\} = \mathbf{F}_{\mathcal{A}}.$$

Let us consider the Moore–Penrose quasi-inverse $A^\dagger$ associated with A. This is an element of $\operatorname{Hom}(\mathbf{F}_{\mathcal{A}}; \mathbf{E}_k)$ satisfying the fundamental identity

$$A \circ A^\dagger = \mathbf{1}_{\operatorname{Im} A} = \mathbf{1}_{\mathbf{F}_{\mathcal{A}}}. \tag{2.6}$$

In particular, by our assumption on f, $A^\dagger f(x)$ is well-defined almost everywhere on Ω and $A^\dagger f \in L^1(\Omega; \mathbf{E}_k)$. Now, we make use of the assumption (b) over each

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E-coordinate to find $u \in BV^k(\Omega; E)$ satisfying

$$D^k u = A^\dagger f\, \mathscr{L}_N + D^j w\,,$$

where $w = \nabla^{k-1} u \in SBV(\Omega; \mathbf{R}^{N^k}_{\mathrm{sym}})$ satisfies the estimate

$$\|u\|_{W^{k-1,1}(\Omega)} + \int_\Omega |\nabla w|\, \mathrm{d}x + \int_{J_w} |[w]|\, \mathrm{d}\mathscr{H}_{N-1} \leqslant C \int_\Omega |A^\dagger f|\, \mathrm{d}x \tag{2.7}$$

for some constant C depending on N, k and $\dim \mathbf{F}_\mathcal{A}$. Here, as usual $D^j w = \tilde{g}\, \mathscr{H}_{N-1} \llcorner J_w$ is the jump part of Dw. Notice that in this case $\tilde{g} = \mathbf{1}_{J_w}[w] \otimes \nu_w$. Pre-composing this expression with A, we get

$$\mathcal{A}u = A[D^k u] = A\left[A^\dagger f\, \mathscr{L}_N + \tilde{g}\, \mathscr{H}_{N-1}\right].$$

Since A is a linear map, we may pull in and distribute A into the densities that belong to the sum of the right-hand side, hence concluding that

$$\mathcal{A}u = \ A \circ A^\dagger f\, \mathscr{L}_N + A\tilde{g}\, \mathscr{H}_{N-1} = f\, \mathscr{L}_N + g\, \mathscr{H}_{N-1},$$

where $g = \mathbf{1}_{J_w} \cdot A([w] \otimes \nu_w)$. In passing to the last equality in the formula above, we have used the almost everywhere pointwise restriction $f(x) \in \mathbf{F}_\mathcal{A}$. Now, we use the fact that at jump points (cf. (2.4)), it holds

$$[w] = \langle [w], \nu_w \rangle_{k-1} \otimes (\nu_w)^{\otimes^{k-1}}.$$

From this we obtain that $g = \mathbf{1}_{J_w} \cdot \mathbb{A}^k(\nu_w)\langle [w],\, \nu_w \rangle_{k-1}$ as desired. Lastly, the bound on the Sobolev norm and total variation of $D^k u$ follow from (2.7), the estimates

$$\int |A^\dagger f|\, \mathrm{d}x \leqslant \|A^\dagger\|_{\mathbf{F} \to \mathbf{E}_k} \int |f|\, \mathrm{d}x,$$

$$\int |g|\, \mathrm{d}\mathscr{H}_{N-1} \leqslant \|A\|_{\mathbf{E}_k \to \mathbf{F}} \int |[w]|\, \mathrm{d}\mathscr{H}_{N-1},$$

and the fact that $\|A^\dagger\|_{\mathbf{F}_\mathcal{A} \to \mathbf{E}_k} = \|A^{-1}\|_{\mathrm{Im}\, A \to \mathbf{F}_\mathcal{A}}$ depends solely on $\mathcal{A}$.[2] This finishes the proof. □

Proof of Theorems 2.1 *and* 2.2. The proof of Theorem 2.1 follows directly from the validity of the Lusin property for gradients ([**1**, Theorem 3]), the previous proposition (with $k = 1$), and the equivalence in Remark 2.3. The proof of Theorem 2.2, on the other hand, follows from the previous proposition (with $k = 2$) and [**15**, Theorem 1.4]. □

[2] This equality of norms follows from (2.6), which implies that the restriction of $A^\dagger$ to $\mathrm{Im}\, A$ is the inverse of the isomorphism $A : (\ker A)^\perp \to \mathrm{Im}\, A$.

An application for normal currents

Let m be a non-negative integer. An *m-dimensional current* T on Ω is an element of the continuous dual $\mathscr{D}_m(\Omega) = \mathscr{D}^m(\Omega)^*$, where

$$\mathscr{D}^m(\Omega) \coloneqq C_c^\infty(\Omega; \wedge^m \mathbf{R}^N)$$

is the space of smooth and compactly supported m-forms on Ω. If $m \geqslant 1$, the (distributional) boundary operator on currents is defined by duality through the rule

$$\partial T \in \mathscr{D}_{m-1}(\Omega)\,, \quad \partial T(\varphi) = T(d\varphi) \text{ whenever } \varphi \in \mathscr{D}^{m-1}(\Omega)\,.$$

Here, d is the exterior derivative operator acting on $\mathscr{D}^m(\Omega)$. The boundary operator then maps $\mathscr{D}_m(\Omega)$ into $\mathscr{D}_{m-1}(\Omega)$. The mass of a current $T \in \mathscr{D}_m(\Omega)$ is defined as

$$\mathbf{M}(T) = \sup\left\{ T(\varphi) : \varphi \in \mathscr{D}^m(\Omega),\ \sup_{x\in\Omega} |\varphi(x)| \leqslant 1 \right\}.$$

A current T is called *normal* if T is representable by a finite Radon measure and either ∂T is representable by a finite Radon measure or $m = 0$. In particular, every normal k-current is a distribution represented by a measure taking values on the space $\wedge_m \mathbf{R}^N = (\wedge^m \mathbf{R}^N)^*$ of m-vectors. The space of normal m-dimensional currents on Ω is denoted by $\mathbf{N}_m(\Omega)$.

Given an element $\xi \in \mathbf{R}^N \cong \wedge_1 \mathbf{R}^N$, we write ξ^* to denote the 1-covector

$$\xi^*(v) \coloneqq \xi \cdot v, \qquad v \in \mathbf{R}^N.$$

The interior multiplication operator $\llcorner\colon \wedge_p \mathbf{R}^N \times \wedge^q \mathbf{R}^N \to \wedge_{p-q} \mathbf{R}^N$ is defined as the adjoint of the exterior multiplication (see e.g. [**14**, Chapter 1.5]):

$$\langle v \llcorner \alpha, \beta \rangle = \langle v, \alpha \wedge \beta \rangle \qquad v \in \wedge_p \mathbf{R}^N,\ \alpha \wedge^q \mathbf{R}^N,\ \beta \in \wedge^{p-q} \mathbf{R}^N.$$

With this notation in mind, we get the following direct application of Theorem 2.1 for the boundary operator acting on normal currents:

Corollary 2.5. *Let $m \in [0, N)$ be an integer and let $S : \Omega \to \wedge_m \mathbf{R}^N$ be an integrable m-vector field. Then, there exists a normal current $T \in \mathbf{N}_{m+1}(\Omega) \cap SBV(\Omega; \wedge_{m+1} \mathbf{R}^N)$ satisfying*

$$\partial T \; = \; S\,\mathscr{L}_N \; + \; [T] \llcorner \nu_T^* \, \mathscr{H}_{N-1} \llcorner J_T$$

Moreover,

$$\mathbf{M}(T) + \mathbf{M}(\partial T) \leqslant C_{m,N} \|S\|_{L^1(\Omega)}.$$

Remark 2.6. A similar result holds for integrable m-forms with $m \in (0, N]$. More precisely, if $\omega \in L^1(\Omega; \wedge^m \mathbf{R}^N)$, then there exists an $(m-1)$-form $\phi \in SBV(\Omega; \wedge^{m-1} \mathbf{R}^N)$ satisfying (see also [**17**, Proposition 2.1])

$$\mathrm{d}\phi = \omega\, \mathcal{L}_N + \nu_\phi^* \wedge [\phi]\, \mathscr{H}_{N-1} \llcorner J_\phi.$$

and

$$\int_\Omega |u| \,\mathrm{d}x + \int_{J_\phi} |[u]| \,\mathrm{d}\mathscr{H}_{N-1} \leqslant C_{m,N} \|\omega\|_{L^1(\Omega)}.$$

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Proof. Since the symbol of the exterior derivative is precisely the exterior multiplication (i.e. $d_m(\xi)\alpha = \xi^* \wedge \alpha$), the symbol of the boundary operator ∂ on $(m+1)$-vectors is precisely (see e.g. [**14**, §4.1.7.])

$$\partial_{m+1}(\xi)e = -(e \llcorner \xi^*), \qquad \xi \in \mathbf{R}^N, e \in \wedge_{m+1}\mathbf{R}^N.$$

In particular, the boundary operator ∂ defines a constant-coefficient first-order operator from $\mathscr{D}_{m+1}(\Omega)$ to $\mathscr{D}_m(\Omega)$. Since (cf. [**14**, §1.5.2])

$$\operatorname{Im}\partial_{m+1}(\xi) = \ker \partial_m(\xi) = \operatorname{span}\{v_1 \wedge \cdots \wedge v_m \mid v_1, \ldots, v_m \in \xi^\perp\},$$

it follows that $\wedge_m \mathbf{R}^N = \operatorname{span}\{\operatorname{Im}\partial_{m+1}(\xi) : |\xi| = 1\} = \mathbf{F}_\partial$. We may thus apply Theorem 1 to find $T \in SBV(\Omega; \wedge_m \mathbf{R}^N)$ satisfying the desired properties: the fact that $T \in \mathbf{N}_{m+1}(\Omega)$ follows directly from the fact that $\mathbf{M}(T) = \|T\|_{L^1(\Omega)}$ and that $\mathbf{M}(\partial T) \leqslant |DT|(\Omega)$. □

Corollary 2.7. *Let $m \in [1, N)$ be an integer and let $S : \Omega \to \wedge_m \mathbf{R}^N$ be an integrable m-vector field. There exists a countably $\mathscr{H}_{N-1}$ rectifiable set J with (oriented) normal ν, and a Borel $(m+1)$-vector field $g : J \to \wedge_{m+1}\mathbf{R}^N$ satisfying*

$$\partial(S\,\mathscr{L}_N - g \llcorner \nu^* \,\mathscr{H}_{N-1} \llcorner J) = 0$$

and

$$\int_J |g| \mathscr{H}_{N-1} \leqslant C \int_\Omega |S| \,\mathrm{d}x.$$

Proof. It is sufficient to use the expression for ∂T in the corollary above and observe that $\partial(\partial T) = 0$. □

The canonical isomorphism $\iota : \mathbf{R}^N \to \wedge_1 \mathbf{R}^N$, identifying vector fields in $\mathbf{R}^N$ with 1-vectors, induces an isometry between vector fields with bounded measure divergence and normal 1-currents. In particular, an exciting and direct application of the previous corollary is the following *rectifiable completion* for systems of vector fields to systems of solenoidal measures.

Corollary 2.8. *If $N \geqslant 2$, then every vector field $\vec{f} \in L^1(\Omega; \mathbf{R}^N)$ is the absolutely continuous part of a solenoidal field measure up to a rectifiable measure. More precisely, there exists a countably $\mathscr{H}_{N-1}$ rectifiable set $\Gamma \subset \Omega$ and a Borel vector field $\vec{a} : \Gamma \to \mathbf{R}^N$ satisfying the differential constraint*

$$\operatorname{div}(\vec{f}\,\mathscr{L}_N + \vec{a}\,\mathscr{H}_{N-1} \llcorner \Gamma) = 0$$

and also the tangential constraint

$$\vec{a}(x) \in \operatorname{Tan}(\Gamma, x) \quad \text{for } \mathscr{H}_{N-1} \text{ almost every } x \in \Gamma.$$

Moreover,

$$\int_\Gamma |\vec{a}| \,\mathrm{d}\mathscr{H}_{N-1} \leqslant C \|\vec{f}\|_{L^1(\Omega)}$$

for a constant C that depends only on N.

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Proof. Applying the assertion of Corollary 2.7 (here we are using that $N \geqslant 2$) to the integrable 1-vector field $\iota\vec{f}$ yields the existence of a countably $\mathscr{H}_{N-1}$ rectifiable set $\Gamma \subset \Omega$ and an integrable 2-vector $g : \Gamma \to \wedge_2 \mathbf{R}^N$ satisfying

$$\partial(\iota\vec{f}\,\mathscr{L}_N - g\llcorner\nu^*\,\mathscr{H}_{N-1}\llcorner\Gamma) = 0$$

and

$$\int_\Gamma |g|\,\mathscr{H}_{N-1} \leqslant C\int_\Omega |f|\,\mathrm{d}x. \tag{2.8}$$

By the identification discussed before, we conclude that

$$\mathrm{div}(\vec{f}\,\mathscr{L}_N + \vec{a}\,\mathscr{H}_{N-1}\llcorner\Gamma) = 0,$$

where $\vec{a} := \iota^{-1}(-g\llcorner\nu^*) : \Gamma \to \mathbf{R}^N$ is integrable on Γ and satisfies the asserted L^1 bounds on Γ. To see the tangential properties of the $\vec{a}$, it suffices to observe that $\vec{a}(x)\cdot\nu(x) = \langle g(x)\llcorner\nu(x)^*,\,\nu(x)^*\rangle = \langle g(x),\,\nu(x)^*\wedge\nu(x)^*\rangle = 0$ for $\mathscr{H}_{N-1}$ almost every $x \in \Gamma$ (on Lebesgue points of g and ν). $\square$

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